Original Article

Modern pension fund diversification

Received (in revised form): 2nd July 2014

Martin Anderson
is Deputy Chief Investment Officer at the Arizona PSPRS Trust where he has responsibility across the portfolio with a focus toward the system's private market/alternative investment vehicles. He is also member of the CFA Society Phoenix and is a CAIA (Chartered Alternative Investment Analyst).

Shan Chen
is a Lead Portfolio Manager at Arizona PSPRS trust where he has responsibility for private and public equity asset classes.

James Hacking
is the Administrator at Arizona PSPRS trust. His four decades of public pension fund service also include the position of Executive Director at the State Universities Retirement System of Illinois, the Minneapolis Employees Retirement Fund, the Public Employees Retirement Association of Minnesota, as well as having served in various capacities with the American Association of Retired Persons.

Marc R. Lieberman
is Chair of the Public Pension/Alternative Investments group of Kutak Rock LLP, where he leads a national team of lawyers dedicated to negotiating alternative investments for pension systems and sovereign wealth funds.

Mark Lundin
is Deputy Chief Investment Officer – Risk at Arizona PSPRS trust where he is responsible for global and portfolio market risk and is Lead Portfolio Manager for Global Tactical Asset Allocation.

Vaida Maleckaite
is Lead Portfolio Manager for Risk Parity at Arizona PSPRS trust, where she is also Investment Operations Analyst responsible for financial and performance reporting, portfolio risk analytics, document management and operation support.

Allan Martin
is a Partner at New England Pension Consultants (NEPC) LLC, where he serves on the Executive Committee and manages their Public Fund Western Region consulting activities. He is also a member of the International Equity Advisory Group, and was recognized in 2012 by aiCIO as one of the top 25 most influential investment consultants.

Ryan Parham
is Chief Investment Officer at Arizona PSPRS trust where he has responsibility for strategic analysis for the PSPRS portfolio. He is also a member of the American Bar Association, member of the Academy of Political Science and admitted to practice before the US Supreme Court.

Mark Steed
is Chief of Staff and also Lead Portfolio Manager at Arizona PSPRS trust, where he has senior responsibility for the Fixed Income and Credit Opportunities portfolios.

Correspondence: Mark Lundin, Arizona PSPRS, 3010 E. Camelback Road, Suite 200, Phoenix, AZ 85016, USA
E-mail: MLundin@psprs.com

ABSTRACT The risk and return characteristics of a highly diversified investment portfolio are examined in an effort to best assess its potential by means that incorporate both conventional risk estimation and performance evaluation. Estimation of performance variability and downside risk often assumes a constant, stable, average covariance matrix of asset returns and only provides an indirect gauge of capacity for the downside compensation interplay between assets. Performance measurement allows for final conclusions to be drawn, but does not capture the structural characteristics leading to results, nor does it make a distinction between chance occurrence and structural bias. The Mahalanobis distance is employed in order to quantify both aspects simultaneously and document a contemporary shift in advanced pension trust management. The asset liability structures of pension trusts allow for unusually long time horizons and managing agencies typically possess the resources necessary to select and maintain opaquely priced investments in a controlled fashion. A particular pension fund history, involving a period of transition from a conventional, strictly US-based mix of stocks, bonds, real estate and cash, to a more diversified set of eight additional asset classes, allows for discussion of first results and assessment of the trend toward diversification.
Journal of Asset Management (2014) **15,** 205–217. doi:10.1057/jam.2014.23

Keywords: Mahalanobis distance; turbulence index; diversification; asset allocation; pension fund; risk

INTRODUCTION

The rational objective of investment is to achieve a return on the investment with the highest practical confidence. The level of desired return is not objective, but subject to a particular investor's goals, constraints, asset, liabilities and a host of third party variables outside the investor's control. Investment involves a connection between a given level of return and the uncertainty that must be born in order to have the potential, but not the guarantee, to achieve desired results. The investor may first set a return goal, or begin by setting a ceiling on acceptable risk and work backwards to consequent expected return. The Capital Asset Pricing Model of Sharpe (1964), Lintner (1965) and Mossin (1966) builds on the mean-variance framework of Modern Portfolio Theory (MPT) of Markowitz (1952) and both hold that a rational investor should not hold diversifiable risk, or systematic risk that goes unrewarded. In practice, diversifying a portfolio typically involves increasing the number of uncorrelated securities in a portfolio to a practical point of diminishing added value. Markowitz (1959) suggested that this limit is approached at the average of the covariance matrix of asset returns.

In the mean-variance framework of MPT, portfolio risk is often characterized by variance and can be written as:

$$\sigma_P^2 = \sum_i \sum_j \omega_i \omega_j \sigma_i \sigma_j \rho_{i,j} \qquad (1)$$

where σ_P is the standard deviation of portfolio returns, σ_i and σ_j are the standard deviations of assets i and j, ω_i and ω_j are the investment weights on assets i and j and $\rho_{i,j}$ is the correlation between returns of assets i and j.

For a portfolio of two assets, this expands to:

$$\sigma_P^2 = \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1\omega_2\sigma_1\sigma_2\rho_{1,2} \qquad (2)$$

If the correlation between these securities is negative, then it is possible to bring portfolio volatility to zero. If correlation is zero, portfolio volatility is the quadrature sum of weighted averages of asset volatilities; an absolute minimum value for which is the smaller of the two asset volatilities divided by the square root of 2. Any positive correlation increases portfolio risk to a level at least above that of the least of the individual asset volatilities. These statements imply that variance, covariance and correlation are unchanging, except by a randomness that cancels out over a long period. Alternative forms of risk quantification have been proposed in many forms but reflect a common skepticism of the idea of purely random deviation from long-term averages as this would appear to reject observations of volatility clustering going back to Mandelbrot (1963) and Fama (1965), and the volatility persistence noted by Bollerslev *et al* (1992), Bollerslev, Engle and Nelson (1994) and Andersen and Bollerslev (1997).

The ambition of risk quantification is to put a number on investment uncertainty. Increasing the number of securities in a portfolio, or investment diversification, can lead to a more stable risk-return profile. When assessed in hindsight, risk most often manifests itself through an average level of uncertainty over many smaller investment periods. The average uncertainty, or risk, has itself an uncertainty that might be interpreted as a variation of variation, or might be expressed as a confidence level or limit. Consideration of risk as the average of variation invariably, however, involves a loss of information that the confidence level does not restore. Chow *et al* (1999) noted this important consequence with respect to Markowitz-style mean-variance optimization. They postulate that asset returns become more volatile and more highly correlated in turbulent markets and the diversification that characterizes the sample on average disappears when it is most needed.

One way to retrieve the information lost by considering an average variation and covariance alone might be to seek higher frequency information within the return period. For example, the uncertainty on monthly risk might be estimated using daily or even intraday pricing information. In practice this is not always convenient, free of complication or even possible. Variance, covariance and correlation manifest themselves differently at different frequencies of information flow. Declines in correlation as return frequency increases (or as return periods decrease) were first observed in the case of stocks as the Epps Effect (Epps, 1979). Subsequently the Epps Effect has been observed in other asset classes as well by Lundin *et al* (1998) and Andersen *et al* (2006 and references therein). More problematic than this, for many assets sufficient information does not exist by any practical means. For example, private equity, special credit opportunity, real estate and real asset investments are interesting alternatives open to the longer time horizon investor that often provide an additional risk premium as compensation for lack of current mark-to-market pricing certainty on shorter time horizons. At the crux of the problem in estimating investment uncertainty is the ability to gauge, for a given investment period, the distance between the return of an investment and the return of an expectation value, or alternatively a target.

In this discussion, we elaborate on a method, advocated by Kritzman and Li (2010), for gauging the distance between a portfolio's return during a finite period of time and some expected return. The utility for this distance, taken over many time periods, to characterize a portfolio and its efficiency is investigated. We apply the Mahalanobis distance to the unique case of a pension trust portfolio that has moved, over a period of 6 years, from a stock/bond/real estate/cash mix of US securities to a vastly broader level of diversification involving 190 separate investment mandates spread over 12 different asset classes. The relation between the Mahalanobis distance and portfolio efficiency is investigated along with its relation to volatility.

THE INSTANTANEOUS AND SCALABLE NATURE OF THE MAHALANOBIS DISTANCE

In a univariate context, the distance between two points, x, is the difference (or absolute value of difference) between their values. This distance is not very informative in a statistical sense and so we speak of distance in terms of standard deviations, σ, which better relate what is a 'usual' distance and what is not:

$$\frac{x-\mu}{\sigma^2} \tag{3}$$

Where μ is a reference point, perhaps the mean of a process, an expectation value, a historical mean or simply zero. In a multivariate setting involving a vector of distances, y, the Euclidean distance could be written as:

$$(y-\mu)^T(y-\mu) \tag{4}$$

This may not be useful for the same reason; there is no accounting for the variances or the covariances of assets' returns which would standardize them for comparison. It would be straightforward to estimate the distance between vectors of asset returns if they were generated by the same process, but this cannot be assumed across either securities or asset classes. Similar to the univariate case, we standardize by the covariance matrix, $\sum$:

$$d^2 = (y-\mu)^T\sum^{-1}(y-\mu) \tag{5}$$

$$d = \sqrt{(y-\mu)^T\sum^{-1}(y-\mu)} \tag{6}$$

Equation (6) was first proposed by Mahalanobis (1927) and is often referred to as the Mahalanobis distance, d. There are two main characteristics of the form of the Mahalanobis distance that make it of particular interest. First, the single-period vector of asset returns, y, give Equation (6) an instantaneous, or point-in-time, character. Second, it effectively standardizes the variance of each variable and covariances between variables to the same scale. The Mahalanobis distance is a way of measuring distance that accounts for correlation between variables, and is related to Hotelling's T^2 distribution. Geometrically, the Mahalanobis distance does this by transforming the data into standardized uncorrelated data and computing the ordinary Euclidean distances for the transformed data. In this sense, the Mahalanobis distance is similar to a univariate z-score; it provides a way to measure distances that takes into account the scale of the data. It is more useful for multinormal distributions, but normality is not a requirement. As a standardized measure, the Mahalanobis distance has no units.

Kritzman and Li (2010) advocate the application of the Mahalanobis distance – in the form of a 'turbulence' index – for stress-testing portfolios, to construct turbulence-resistant portfolios and to scale exposure to risk in order to improve performance. Our interpretation of their turbulence index definition, being the square of the Mahalanobis distance in Equation (5), is that it emphasizes large deviations in single periods over small deviations in many, and also as a way to achieve a favorable convex property if applied in the context of a minimization problem. We perceive that this may be appropriate, depending on the application, but that nevertheless their interpretations remain equally as valid for both alternatives. Therefore we will also refer to the Mahalanobis distance as 'turbulence' in order to make a distinction from volatility.

The Mahalanobis distance measures the deviation of a particular sampling from the centroid (multidimensional mean) of a distribution, where the distribution is characterized by a mean and covariance matrix. As with a culinary recipe, in which the importance of any deviations in specific ingredient amounts may vary widely as measured by the palate, the temporary deviation of portfolio component returns from their distributional norms of means and covariance over different time periods may produce widely different portfolio returns after combination through weighted asset allocation.

The use of the inverse of the covariance matrix in the Mahalanobis distance has the effect of standardizing all variables to the same variance and eliminating correlations. If a variable has a larger variance than another, it receives relatively less weight in a Mahalanobis distance. Similarly, two highly correlated variables do not contribute as much as two variables that are less correlated. When there is no correlation, the inverse matrix is a unit matrix and the Mahalanobis distance is equal to Euclidean distance.

If the covariance matrix is positive definite then the Mahalanobis distance squared will be positive, regardless of the value of $x-\mu$. So long as the covariance matrix is non-singular with a non-zero determinant, it can be inverted but may produce imaginary Mahalanobis distances.

In the discussion that follows, the Mahalanobis distance is applied to returns delivered by asset classes over the history of a pension fund with an interesting history involving a 6-year campaign of diversification. The pension fund used for empirical results consists of sub-portfolios that have been labeled with commonly known assets class titles, though it would not be uncommon to find another agent using precisely the same labels to describe something rather different. Just as one institution may fill an asset class according to a slightly different definition, good versus poor security selection may define an asset class more than a title. In addition, sub-portfolios, or asset classes, in this discussion consist largely of investment mandates. The mandates in turn hold securities. We find no fundamental reason to differentiate between securities, group of securities or asset classes as portfolio components, presuming that their weight, performance and covariance are descriptive.

EXAMINATION OF A PARTICULAR PENSION TRUST UNDER A BROAD DIVERSIFICATION PROGRAM

The Arizona Public Service Personnel Retirement System (PSPRS), a USD-based pension Trust was founded in 1968. It initially involved itself only in direct investment in US securities; a mix of stocks, bonds, real estate (added in 1992) and cash. Investment management consisted of active security selection within each asset class. In 2007 it made its first international equity investments. Non-USD investment returns were unhedged and brought direct foreign exchange risk into the portfolio. In 2008, the assets of the PSPRS Trust were commingled with those of two other pension systems managed by a unitary board of trustees with the assets of all three systems merged into the Arizona PSPRS Trust whose mandate was to invest in a broad variety of assets including many uncorrelated to the publicly traded equity and bond markets. Seven additional asset classes were introduced in the period following 2008, making a total of 12 by 2012. Since 2007, changes in asset allocation and asset class benchmarks were made annually. This evolution required considerable planning, study and legislation in order to make the Trust's present form possible, as is discussed further in Lieberman and Lundin (2014).

As of December 2013, Arizona PSPRS Trust controlled US$7.8 billion in assets under management and asset classes included:

- *Public equity (US listed stocks since inception)*: Investments in public equities broadly diversified to include large-cap, mid-cap and small-cap exposure. The public equity program utilizes various strategies, including active management, passive management, long/short, enhanced index, portable alpha and hybrid active.
- *US fixed income (US securities since inception)*: Focuses on marketability and liquidity

while investing directly, or through funds, in government and corporate credit.

- *Cash (Since inception)*: Earning close to the risk-free overnight interest rate.
- *Real estate (US real estate introduced in 1992)*: Investments that are in the commercial sectors of office, retail and industrial (including logistics) and the leased (rather than owner occupied) residential real estate sector, as well as residential lots. The various strategies in the real estate program include core income-producing, value add, opportunistic, real estate mortgage credit and residential. Non-US real estate investments were added in June of 2008, but not at levels significant enough to justify consideration as a separate asset class.
- *Non-US based Equity (Introduced July of 2007)*: Investments in non-US public equities broadly diversified to include large-cap, mid-cap and small-cap exposure in both developed and emerging markets.
- *Private equity (Introduced in July 2008)*: Investments in equity or debt (with equity participation) in commingled assets that are generally not traded on public exchanges and usually illiquid in nature.
- *Credit opportunities (Introduced in July 2008)*: Investments in corporate credit including bank loans, high yield debt, convertible securities, 'distressed' corporate debt, mezzanine loans, structured products (such as CLOs and MBS) and other credit-sensitive instruments that may or may not be publicly listed.
- *Real assets (Introduced in April 2009)*: Investments in energy, core capital assets, special situations, natural resources, infrastructure, commodities and marketable securities.
- *Global tactical asset allocation (Introduced in March 2010)*: Encompasses strategies that trade across highly diversified and liquid markets utilizing distinct processes to execute their broadly global trading strategies. Strategies within this sub-portfolio include Global Tactical Asset Allocation, Commodity Trading Advisor, Global Macro and Multi Asset Strategies.
- *Non-US based fixed income (Introduced in November 2010)*: Government and corporate credit in both non-US developed and emerging markets.
- *Absolute return (Introduced in November 2010)*: Consists of management by general partners that do not adhere to specific benchmarks and whose strategies do not neatly fit within another asset classes.
- *Risk parity (Introduced in July 2012)*: Allocation of capital among major asset classes such as equity, nominal bonds (such as the US Treasuries), inflation linked bonds (such as TIPS) and commodities, based on expected volatility, to capture risk premiums across asset classes.

The Mahalanobis distance as defined by Equation (6) was estimated for the Arizona PSPRS Trust using 11 years of weighted asset class monthly returns from January 2003 to December of 2013 and results are shown in Figure 1 as darkened bars. For comparison as a benchmark and as a proxy 'market', a portfolio mix of international stocks (65 per cent), international bonds (40 per cent) and cash (5 per cent) was also estimated and is shown as the white bars in Figure 1. The MSCI World Net Total Return Index was used to represent stocks, The Barclays Global Aggregate Index was used to represent bonds and 3 month US treasury bills represented cash. The monthly vector of PSPRS asset class returns were weighted by asset class using known beginning of month values, and were net of fees, whereas the monthly returns of the comparison portfolio were treated as though perfectly rebalanced, or maintained at a constant 65/35/5 per cent stock/bond/cash mix. Elements of the covariance matrix and a vector of mean return values used in Equation (6) were estimated using the entire period available for each asset class during the 11 year total period. Results might therefore be considered as in-sample.

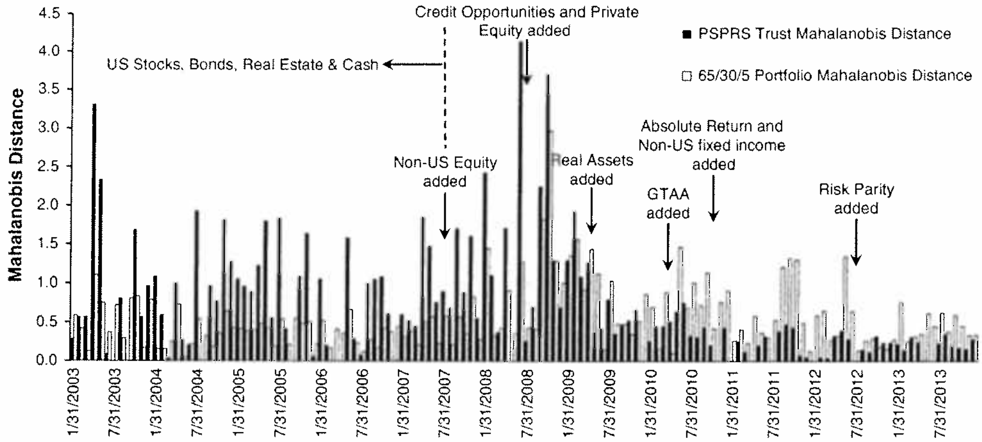

Figure 1: Monthly Mahalanobis distances estimated for the PSPRS Trust (darkened bars) and for a comparison or market proxy benchmark portfolio (clear bars). Also shown are dates of inception of new asset classes. A significant peak in distances is noted in and around the period of market turmoil that occurred during the second half of 2008 and Q1 of 2009. Overall, reduction in the average distance for the case of the PSPRS portfolio (darkened bars) is noted moving forward through the 11 year sample of data.

The points of introduction of new asset classes to the pension are also indicated in Figure 1. The covariance matrix for the 65/35/5 portfolio was positive definite, and all computed distances were real numbers. However, the covariance matrix for the Arizona PSPRS Trust portfolio was not positive definite and 17 of the 132 monthly Mahalanobis distances in the 11-year PSPRS data sample were imaginary numbers. The imaginary distances in the PSPRS sample did not appear to occur with significant clustering in time, though there were two cases of consecutive months with imaginary distances, March/April of 2006 and May/June of 2013. Of the imaginary distances observed, the minimum, average and maximum were $d_{min} = 0.096i$, $d_{avg} = 0.418i$ and $d_{max} = 2.213i$. A number of methods exist for forcing a covariance matrix to be positive definite. However, for the purposes of this discussion we elected to avoid the potential for introducing an artificial bias in the data and opted instead to exclude the 17 observed imaginary distances from all analysis.

It is evident from the figure that the Mahalanobis distance for the pension fund (the darkened bars in Figure 1) begins higher and reduces over time as new asset classes are introduced and the monthly Mahalanobis distances are shown as a function of number of asset classes in Figure 2.

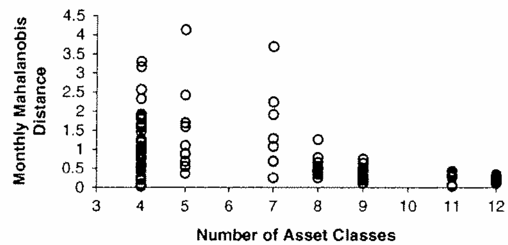

Figure 2: Monthly Mahalanobis distances reported as a function of the number of asset classes in the PSPRS portfolio. Sixth and seventh asset classes were added simultaneously as were tenth and eleventh, resulting in gaps. A significant peak in both maximum distances and average distances is noted with the addition of sixth and seventh asset classes and this is directly attributable to a marked increase in overall portfolio volatility during the second half of 2008.

Figure 2 gives an indication, both in terms of mean and peak Mahalanobis distance, of an inverse dependence on the number of assets. We note that the apparent peak in Figure 2, on the addition of a fifth asset class, non-US equities in July of 2007, is the result of two outlying months; the greatest distance (4.1) in the five-asset data group occurred in June of 2008 as the Arizona PSPRS Trust lost 10 per cent on US and non-US equities combined.

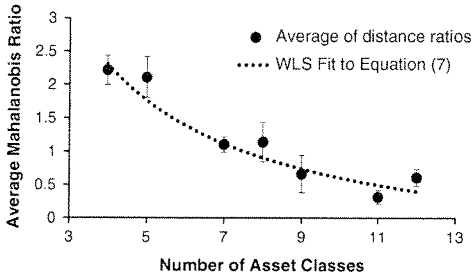

Figure 3: Categorized by the number of asset classes in the portfolio, the average of the ratios of the Arizona PSPRS Mahalanobis distances standardized by 60/35/5 comparison portfolio is shown along with standard errors estimated as the standard deviation of distance ratios divided by the square root of the number of observations used for the ratio average. Also shown as the dotted line is the result of a Weighted Least Squares (WLS) fit of Equation (7) to the data.

The average value of distances for the period involving five asset classes is higher, but not unusually so. However, following June of 2008, there are an unusual number of months with larger distances, and a significantly large average value of distances, with the addition of the sixth and seventh asset classes (Credit Opportunities and Private Equity, simultaneously in July of 2008). This is not surprising in view of the dramatically volatile market conditions during the second half of 2008. During that period the PSPRS portfolio as a whole suffered unusual losses, as did most other pension and non-pension investment funds. However, these observations raise the issue of volatility of financial markets as a whole when using the Mahalanobis distance to assess a particular portfolio's performance.

In an attempt to remove the effect of general market turbulence on portfolio turbulence, the Arizona PSPRS Trust Mahalanobis distances were normalized by the distances of the stock/bond/cash comparison portfolio and the average of that ratio, classified according to the number of asset classes is shown in Figure 3. Accompanying each ratio in Figure 3 is an estimation of its standard error, taken to be the standard deviation of ratios used to estimate the average, divided by the square root of the number of monthly observations.

Also shown in Figure 3 is the result of a Weighted Least Squares fit, which takes into account the sampling errors shown, of Equation (7) to the Average Mahalanobis Ratio (AMR) data.

$$\mathrm{AMR} = c_1 + \frac{c_2}{\sqrt{n}} \qquad (7)$$

where c_1 and c_2 are fit constants and n is the number of asset classes involved. The result of the fit ($c_1 = -2.163$ and $c_2 = 8.778$) is shown as a dotted line and the goodness of fit R^2 was 0.933. The result of a χ^2 test was 0.9993. The form of the dependence – the inverse square root of the number of asset classes – is consistent with what is predicted by Markowitz (1959) and Elton and Gruber (1977) for the reduction of unsystematic risk as the result of diversification. We note that this discussion treats asset classes as assets and in practice we find no fundamental reason not to do so. The pension fund under study consisted of asset class sub-portfolios, which consist largely of mandates, which in turn consist of securities. Risk may be aggregated on any of these levels provided that weights and covariance are consistent and sufficiently descriptive.

In order to further address the issue of background or general market turbulence as a function of time, the Mahalanobis distances for both the pension trust portfolio and the stock/bond/cash comparison portfolio were tabulated for each year from 2003 to 2013, irrespective of the number of asset classes implemented. Results are shown in Figure 4.

The dramatic increase in the Mahalanobis distance, for both the PSPRS Trust and for the comparison portfolio is evident in 2008, marking the beginning of the Great Recession. Judging by the 65/35/5 comparison portfolio, this peak is followed by a gradual decline toward pre-2008 levels in following years. In an attempt to remove the influence of general market volatility from PSPRS Trust results, the average distances of

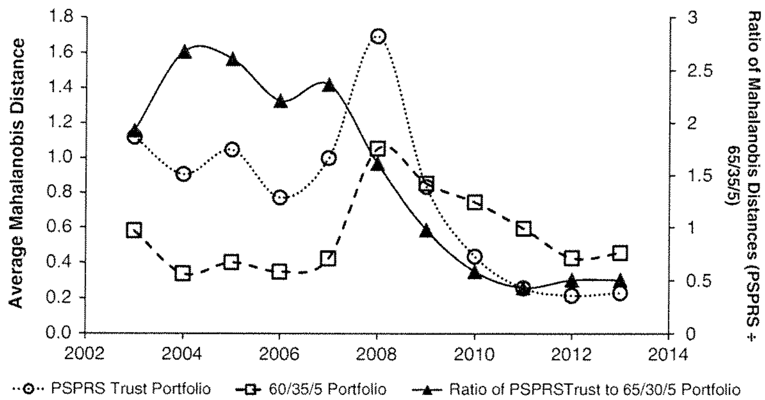

Figure 4: The average of monthly Mahalanobis distances reported by year for the PSPRS portfolio (clear circles, dotted line and left scale) and a comparative, market proxy portfolio (clear squares, dashed line and left scale) mix of international stocks (65 per cent), bonds (35 per cent) and cash (5 per cent). In addition, the ratio of the yearly average of PSPRS to the comparison portfolio is shown (darkened triangles, solid line and right scale). The lines shown are smooth polynomial functions and serve only to guide the eye. From 2003 to 2008 the PSPRS portfolio delivered a factor of 2.4 greater distances on average than the comparison portfolio. Average distance was equal to the comparison portfolio during 2009 and was roughly half the average distance of the comparison portfolio in the period from 2010 to 2013.

the PSPRS Trust (the clear circles, dotted line and left scale of Figure 4) were normalized by those of the comparison portfolio (clear squares, dotted line and left scale of Figure 4) and resulting ratios are shown as the darkened triangles, solid line (corresponding to the right scale) in Figure 4.

A number of observations are readily apparent from Figure 4. Turbulence of the PSPRS trust was a factor of 2.4 times greater, on average, than that of the 60/35/5 comparison portfolio in the period from 2003 to 2007. Examination of this period reveals a number of influences that contributed to this large factor difference. For the case of the PSPRS trust portfolio, there existed a higher concentration of stocks; 68.4 per cent, on average, compared with 60 per cent for the 60/35/5 portfolio. In addition, significant stock selection activity led to name concentrations, and these typically involved higher than average volatility public securities, while the bond portfolio was less diversified than the Barclays Global Aggregate Index used in the 60/35/5 portfolio. Finally, real estate holdings, whose asset values were marked-to-market less frequently (which may have resulted in a volatility reducing, price smoothing), tended to induce significant price change shocks during that period. Until July of 2007, when the fifth asset class (non-US equities) was added to the portfolio, the PSPRS Trust portfolio was static in terms of asset allocation and by no means similar to the comparison portfolio. In addition to holding only US securities (the comparison portfolio consists of international stocks and bonds), the PSPRS portfolio also held direct real estate investments and involved a material level of active management through security selection. One could argue that 2008 and 2009 were transition years, with turbulence of the Arizona PSPRS Trust portfolio declining markedly in both absolute terms and relative to the comparison portfolio. By 2009, the turbulence of both portfolios was roughly equal. Beyond 2009, in the period from 2010 to 2013, turbulence of the PSPRS portfolio became half (0.51 times) that of the comparison portfolio, on average. It is a fact that in the period from 2007 to 2008, the number of asset classes in the PSRPS portfolio was doubled and as each was introduced it was built-out and grew more diversified in its own right over time. That the campaign of increased diversification did have an effect is evident in both absolute and relative terms. Compared with the 65/35/5 stock/bond/mix portfolio, which to at least some extent serves as a suitable 'market' proxy, the Arizona PSPRS Trust has become 4.7 times less turbulent since the period before the campaign of broader asset allocation began.

MAHALANOBIS DISTANCE AND PORTFOLIO EFFICIENCY

A less volatile, or less turbulent, portfolio is not necessarily an indication of a more efficient (or 'better') portfolio and a distinction is noted between portfolio de-risking (or moving to less volatile assets) and diversification, which we interpret as degree of portfolio efficiency. For the purposes of this discussion, we define maximum efficiency as the minimum risk for a given level of return, or alternatively, the maximum return for a given level of risk. The portfolio efficiency of the Arizona PSPRS Trust was estimated in terms of the Sharpe ratio, or the ratio of return above the risk-free interest rate divided by the risk of achieving that return. A rolling 1-year window was applied, during which the numerator of the Sharpe ratio consisted of the previous 12 months of returns above the risk-free interest rate. The denominator consisted of the annualized standard deviation of these returns and results are shown in Figure 5.

The influence of a significant increase in diversification since 2007 is apparent in Figure 5, with the exception of significantly negative results in the period of the second half of 2008 through the first quarter of 2009; the Great Recession. The −29.7 per cent loss during that 9-month period was not negative in relative terms. The MSCI World Index measured in USD, for example, had a performance of −53.8 per cent during the same period. Relatively speaking, the Arizona PSPRS Trust portfolio has improved markedly since 2009. New England Pension Consultants (NEPC, 2014) compared the 3 year PSPRS Sharpe ratio with other pensions trusts for which it holds information and found that from the period 2009 to 2013 the pension fund moved from being a bottom quartile performer to the top decile in a sampling of 56 peers. The information in Figure 5 is a smoothed curve derived from 1-year moving windows of monthly information used to estimate Sharpe ratios and should not be interpreted as a performance statement. In addition to the short-term horizon of 1 year consisting of 12 monthly observations, the curve is not constructed from independent, but rather overlapping information. However, we feel that this treatment provides the granularity desired in the context of this discussion which interests itself in the structure of volatility and turbulence and is useful as a general guide of portfolio performance evolution.

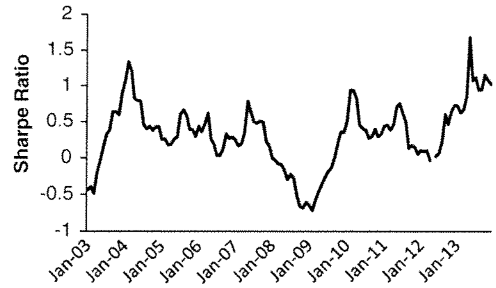

Figure 5: Sharpe ratios estimated in 1-year moving windows of monthly returns of the Arizona PSPRS trust portfolio for the period from 2003 to 2013. Notable negative performance is noted during the second half of 2008 and first quarter of 2009, after which Sharpe ratios improved drastically from 2009 to 2013.

Mahalanobis distances for each month are shown as a function of Sharpe ratios (estimated monthly, but that reflect performance of the previous year) are shown in Figure 6.

While the pension Arizona PSPRS Trust has decreased by a factor of four in turbulence normalized to the market (Figure 4), there also appears to be an inverse relation between the Mahalanobis distance and portfolio efficiency as measured by the Sharpe ratio, as shown in Figure 6, which includes no normalization to the comparison portfolio.

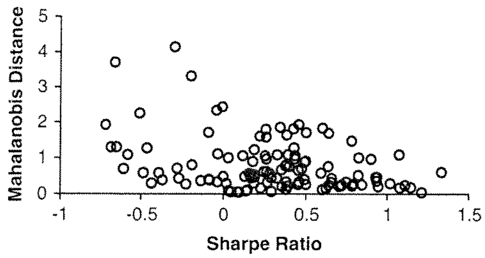

Figure 6: Monthly PSPRS Mahalanobis distances as a function of corresponding PSPRS monthly Sharpe ratios computed over the preceding year using overlapping intervals.

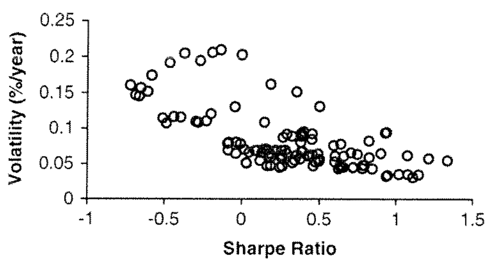

Figure 7: PSPRS annualized standard deviation of monthly returns as a function of corresponding PSPRS monthly Sharpe ratios computed over the preceding year using overlapping intervals.

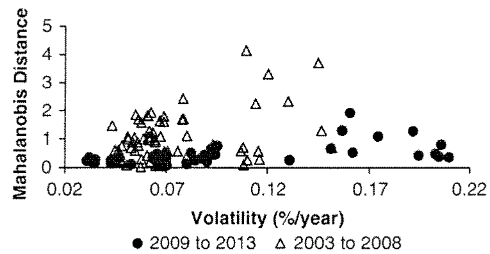

Figure 8: PSPRS monthly Mahalanobis distances shown as a function of one moving windows of annualized standard deviations of PSPRS monthly returns from 2003 to 2013. The discrimination potential of the distance is noted, in this case between the less diversified portfolio implemented between 2003 and 2008 (clear triangles) and the more diversified portfolio (darkened circles) from 2009 to 2013.

We interpret this to be an indication that both the Sharpe ratio and the variability of the Sharpe ratio decrease at lower levels of turbulence. However, a question remains as to whether Sharpe ratios increased simply as a result of reduced portfolio volatility alone, at a similar rate of return. The Mahalanobis distance is shown as a function of Sharpe ratio in Figure 7.

Volatility in Figure 7 was estimated as the standard deviation of portfolio returns using a 1-year moving window, similar to the manner in which Sharpe ratios were estimated. From the figure, one could draw similar conclusions regarding the influence of portfolio volatility on portfolio efficiency as with the Mahalanobis distance, or turbulence. As related in Figure 5, the pension trust moved from the bottom quartile to the top decile among peers. While Figure 5 and the generally inverse relations of Figures 6 and 7 provide evidence that the Arizona PSPRS Trust portfolio under study became significantly more efficient, as defined by the Sharpe ratio, during the total period under study existing information does not allow the conclusion that a lower Mahalanobis distance has any more direct link to portfolio efficiency than does volatility.

MAHALANOBIS DISTANCE AND VOLATILITY

We perceive the Mahalanobis distance to have an advantage over the standard deviation of portfolio returns through its potential to gauge risk at a particular point in time, as opposed to an average taken through a historical period, though we note that both measure variation (or normalized distance) about either zero or some historical mean value. Figure 8 shows estimated monthly Mahalanobis distances as a function of 1-year moving windows of return standard deviations. As both rely on the covariance matrix of returns, it is not surprising that a loose relationship is apparent. The Mahalanobis distance was initially conceived as a discrimination tool and, operating under the assumption that the pension fund portfolio significantly adapted, before compared with after diversification, the figure makes a distinction between distances estimated for the period

2003–2008 (clear triangles) and 2009–2013 (darkened circles).

A linear regression of all (2003–2013, all data shown in Figure 8) monthly Mahalanobis distances to the standard deviations of monthly portfolio returns for the year prior was performed and resulted in a dependence, or slope value of 0.26, with an unconvincing R^2 of 0.33. The data sample was limited and involved overlapping intervals that presumably lead to information overlap between the two samples shown separately in Figure 8. However, we do note some difference in portfolio character apparent between the two periods, making the discriminatory potential of the Mahalanobis distance apparent and an interesting avenue for future research.

CONCLUSIONS

Performance results from a pension trust portfolio that has evolved from a US-based, stock/bond/cash mix to a highly diversified international investment vehicle have been used in order to survey the potential of Mahalanobis distances. Through diversification, the Arizona PSPRS pension Trust portfolio evolved from being 2.4 times more turbulent than a 65/35/5 per cent international stock/bond/cash mix to one that is now half as turbulent as the same benchmark. The unique history of this pension fund, involving a factor of 4.7 times reduction in turbulence, has facilitated a study of the Mahalanobis distance as a measure of portfolio volatility, diversification and efficiency.

Empirical results are compatible with an inverse square root relation between risk, as captured by the Mahalanobis distance, and the number of assets in a portfolio. As with volatility, the Mahalanobis distance for a particular portfolio is dependent, in part, on a background of general market conditions and a first demonstration revealed that scaling Mahalanobis distances of a portfolio to the distances of a relevant benchmark provides relevant information regarding the evolution of investment management. Not surprisingly, in view of its origins, we perceive that the Mahalanobis distance allows for useful relative portfolio comparisons, characterization and discrimination. There are indications that a portfolio with a lower Mahalanobis distance may be more efficient than one with a higher Mahalanobis distance, but further investigation is required in order to draw firm conclusions.

An important distinction between volatility and turbulence as measured by the Mahalanobis distance is that the latter can be more directly and instantaneously observed for a particular investment period without the need for intraperiod price changes. The interpretation and implications of an imaginary Mahalanobis distance and the effect of forcing the portfolio covariance matrix to be positive definite should be further investigated. We have used the simulated returns from an international portfolio of stock, bond and cash indices as both a comparison portfolio and 'market' proxy. First results regarding standardization of the Mahalanobis distance are promising and further study might be dedicated to determining an appropriate benchmark. Given the preliminary conclusions we have drawn, next steps include investigation of the potential for use of the Mahalanobis distance in mean-variance style efficient frontier portfolio optimization.

REFERENCES

Andersen, T.G. and Bollerslev, T. (1997) Answering the Critics: Yes, ARCH Models Do Provide Good Volatility Forecasts, NBER Working Paper No. 6023, April.

Andersen, T.G., Bollerslev, T., Christoffersen, P.F. and Diebold, F.X. (2006) Volatility and correlation forecasting. In: G. Elliott, C.W.J. Granger and A. Timmermann (eds.) *Handbook of Economic Forecasting*. Vol. 1, Amsterdam, the Netherlands: Elsevier, pp. 779–792.

Bollerslev, T., Chou, R.Y and Kroner, K.F. (1992) ARCH modeling in finance: A review of the theory and empirical evidence. *Journal of Econometrics* 52(1–2): 5–59.

Bollerslev, T., Engle, R.F and Nelson, D.B. (1994) ARCH Models. In: R.F. Engle and D. McFadden (eds.) *Handbook of Econometrics*. Vol. IV, Amsterdam: North Holland.

Chow, G., Jacquier, E., Lowrey, K. and Kritzman, M. (1999) Optimal portfolios in good times and bad. *Financial Analysts Journal* 55(3)(May/June): 65–73.

Elton, E.J. and Gruber, M.J. (1977) Risk reduction and portfolio size: An analytical solution. *Journal of Business* 50(4) October: 415–437.

Epps, T. (1979) Comovements in stock prices in the very short run. *Journal of the American Statistical Association* 74(366): 291–298.

Fama, E. (1965) The behavior of stock market prices. *Journal of Business* 38(1): 34–105.

Kritzman, M. and Li, Y. (2010) Skulls, financial turbulence, and risk management. *Financial Analysts Journal* 66(5): 30–41.

Lieberman, M. and Lundin, M. (2014) From sow's ear to silk purse: The Transformation of the Arizona Public Safety Personnel Retirement System, *Benefits Magazine*, June 2014, pp. 2–7.

Lintner, J. (1965) The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *Review of Economics and Statistics* 47(1): 13–37.

Lundin, M., Dacorogna, M.M. and Muller, U. (1998) Correlation of high frequency financial time series. In: P. Lequeux (ed.) *The Financial Markets Tick by Tick*, London: Wiley, pp. 91–126.

Mahalanobis, P.C. (1927) Analysis of race-mixture in Bengal. *Journal of the Asiatic Society of Bengal* 23: 301–333.

Mandelbrot, B.B (1963) Volatility Estimation Using Extreme Values of Asset Prices, Manuscript, Olin School of Business, Washington University.

Markowitz, H. (1952) Portfolio selection. *Journal of Finance* 7(1) March: 77–91.

Markowitz, H. (1959) *Portfolio Selection: Efficient Diversification of Investments*, New York: John Wiley & Sons, p. 111.

Mossin, J. (1966) Equilibrium in a capital asset market. *Econometrica* 34(4): 768–783.

New England Pension Consultants, LLC (2014) *Private Communication,* New England Pension Consultants, LLC.

Sharpe, W. F. (1964) Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance* 19(3): 425–442.